

## Section 11.8 Power Series

A power series is of the form  $\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \dots$  where  $x$  is a variable and the  $C_n$ 's are constants, called the coefficients of the series. For each value of  $x$ ,  $\sum C_n x^n$  is a series of constants which we can test for convergence or divergence.

The series  $\sum C_n x^n$  may converge for some values of  $x$ , and diverge for others. The sum of the series is a function  $f(x) = C_0 + C_1 x + C_2 x^2 + \dots$ ; the domain of  $f$  is all the values of  $x$  for which  $\sum C_n x^n$  converges.

Examples ① If all  $C_n$ 's are equal to 1, we have the geometric Power Series  $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$  when  $|x| < 1$ , i.e. for  $-1 < x < 1$ .

More generally, a series of the form  $\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \dots$  is called "a power series in  $(x-a)$ " or "a power series centered at  $x=a$ ".

Ex ② For what values of  $x$  is the series  $\sum_{n=0}^{\infty} n! x^n$  convergent? Let's use the ratio test with  $a_n = n! x^n$ . we want  $x$  such that  $\lim \left| \frac{a_{n+1}}{a_n} \right| < 1$ .

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| = |x| \cdot \lim_{n \rightarrow \infty} |(n+1)| = \begin{cases} 0 < 1 & \text{if } x=0 \\ \infty > 1 & \text{if } x \neq 0 \end{cases}$ . By the ratio test the series converges for  $x=0$ , diverges otherwise.

Ex ③ For what values of  $x$  does  $\sum_{n=0}^{\infty} \frac{(x-3)^n}{3^n}$  converge? let  $a_n = \frac{(x-3)^n}{3^n}$ . Then

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x-3)^n} \right| = |x-3| \cdot \lim_{n \rightarrow \infty} \left| \frac{3^n}{3^{n+1}} \right| = |x-3| < 1$  for  $-1 < x-3 < 1$  or  $2 < x < 4$ . Series converges in the "open interval of conv."

$(2, 4)$  and diverges outside this interval. But what about the endpoints of the interval, namely,  $x=2$  and  $x=4$ ?

At  $x=2$ ,  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |2-3| = 1$ ; at  $x=4$ ,  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = |4-3| = 1$ . So the ratio test fails at the endpoints. Instead, we test these points separately:

• at  $x=2$ ,  $\sum \frac{(x-3)^n}{3^n} = \sum \frac{(-1)^n}{3^n}$  converges by the Alternating series test.

• at  $x=4$ ,  $\sum \frac{(x-3)^n}{3^n} = \sum \frac{1}{3^n}$  diverges by p-series,  $p=1$ .

Therefore the power series converges for  $x \in [2, 4)$  and diverges elsewhere.

Ex (4) Find the domain of the Bessel function of order 0, defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}. \quad \text{Let } a_n = \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}. \quad \text{Then } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2} \cdot 2^{2n} (n!)^2}{(-1)^n x^{2n} \cdot 2^{2n+2} (n+1)!^2} \right|$$

$$= \lim_{n \rightarrow \infty} \left| x^2 \frac{1}{4(n+1)^2} \right| = |x^2| \lim_{n \rightarrow \infty} \left| \frac{1}{4(n+1)^2} \right| = 0 < 1 \text{ for all real numbers } x.$$

This means that the series  $J_0(x)$  converges for all  $x$ ; the domain of  $J_0$  is therefore  $(-\infty, \infty)$ . Let's summarize the 3 examples above in a theorem:

theorem: For a given power series, there are only three possibilities

(1) the series converges only at the center ( $x=a$ ) (ex:  $\sum n! x^n$ )

(2) the series converges for all  $x$  (ex:  $J_0(x)$ ).

(3) there is a positive number  $R$  such that the series converges if  $|x-a| < R$  and diverges if  $|x-a| > R$  (ex:  $\sum \frac{(x-3)^n}{3^n}$ ;  $|x-3| < 1 \Rightarrow R=1, a=3$ ).

The number  $R$  is called the radius of convergence; the interval

$(a-R, a+R)$  (i.e. where  $|x-a| < R$ ) is the open interval of convergence.

In Case (1) of the theorem,  $R=0$ ,  $(a-R, a+R) = \{0\}$ ;

In Case (2) of the theorem,  $R=\infty$ ,  $(a-R, a+R) = (-\infty, \infty)$ .

Ex ⑤ Find the radius and interval of convergence of  $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$ .

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right| = |3x| \cdot \lim_{n \rightarrow \infty} \left| \frac{\sqrt{n+1}}{\sqrt{n+2}} \right| = |3x|.$$

Now,  $|3x| < 1 \Rightarrow |x| < \frac{1}{3} \Rightarrow R = \frac{1}{3}$ . The open interval of conv. is  $(-\frac{1}{3}, \frac{1}{3})$ . Let us now test the endpoints  $x = \pm \frac{1}{3}$ .

•  $x = -\frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$  diverges by L.C.T to  $\sum \frac{1}{n^{1/2}}$ .

•  $x = \frac{1}{3} \Rightarrow \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$  Converges by the Alternating Series test.

Therefore, the interval of convergence is  $(-\frac{1}{3}, \frac{1}{3}]$ .

Ex ⑥ Find the radius and interval of convergence of  $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ .

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} \right| = \left| \frac{x+2}{3} \right|. \quad \left| \frac{x+2}{3} \right| < 1 \Rightarrow |x+2| < 3.$$

This means that the radius of convergence is 3. Here the center is  $x = -2$ .

Now  $|x+2| < 3 \Rightarrow -5 < x < 1$ ; let's test the endpoints:

•  $x = -5 \Rightarrow \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{3}$  diverges by the  $n^{\text{th}}$  term test for divergence.

•  $x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{n \cdot 3^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n}{3}$  diverges for the same reason.

Therefore, the interval of convergence is  $(-5, 1)$ .

Ex ⑦  $\sum_{n=1}^{\infty} \frac{(2x-1)^{2n+1}}{n^{3/2}}$ .  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(2x-1)^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{(2x-1)^{2n+1}} \right| = (2x-1)^2$ .

Now,  $(2x-1)^2 < 1 \Rightarrow -1 < 2x-1 < 1 \Rightarrow -\frac{1}{2} < x - \frac{1}{2} < \frac{1}{2}$ ; this means that

the center is  $a = \frac{1}{2}$ , and the radius of conv. is  $R = \frac{1}{2} \Rightarrow 0 < x < 1$

•  $x = 0 \Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{-1}{n^{3/2}}$  Conv. by p-Series,  $p = \frac{3}{2} > 1$

•  $x = 1 \Rightarrow \sum_{n=0}^{\infty} \frac{1^{2n+1}}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$  Conv. by p-Series,  $p = \frac{3}{2} > 1$ .

Therefore, the interval of convergence is  $[0, 1]$ .